Tensor Potential Description of Matter and Space, II Semi-guage and the Law of Conservation.

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Abstract:

Considered a unified field theory approach describing matter and space (metric tensor) by means of a 3-index tensor P^i_{jk}. It is shown that if the Lagrangian has partial U(1) gauge (semi-gauge) of the type $P^i_{jk} - > P^i_{jk} + a\delta^i_j\phi_{,k} + b\delta^i_k\phi_{,j} + cg_{jk}g^{mi}\phi_{,m}$ where a,b,c are constants, then the Euler equations of motion contain the covariant low of conservation in the form $J^k_{;k} = 0$.

Law of Conservation

In the first paper [1] on this subject we proposed a concept of describing the matter (including the gravitational matter) and the space by means of one entity - 3-index tensor (called tensor potential) P^i_{jk} with no a priori set symmetries. The metric tensor g_{ij} is defined as a quadratic function of P^i_{jk}.

$$g_{ij} = b_1\bar{P}^m_{ni}\bar{P}^n_{mj} + b_2\hat{P}^m_{in}\hat{P}^n_{jm} + b_3(\bar{P}^m_{in}\hat{P}^n_{mj} + \bar{P}^m_{jn}\hat{P}^n_{mi}) + b_4\bar{P}_m\bar{P}^m_{ij} + b_5\hat{P}_m\hat{P}^m_{ij} + b_6\bar{P}_i\bar{P}_j + b_7\hat{P}_i\hat{P}_j + b_8(\bar{P}_i\hat{P}_j + \bar{P}_j\hat{P}_i) \qquad (1)$$

In the above equation, $\bar{P}^i_{jk}$ and $\hat{P}^i_{jk}$ are the symmetric and antisymmetric (in low indices) parts of the tensor P^i_{jk}, respectively. Vectors $\bar{P}_k = \bar{P}^i_{ik}$ and $\hat{P}k = \hat{P}^i_{jk}$ are the vectors obtained by contraction of the tensors $\bar{P}^i_{jk}$ and $\hat{P}^i_{jk}$. The coefficients b_1 thru b_8 are constants.

In [1] we suggested that g_{ij} depends only on symmetric (low indices) part of P^i_{jk} with these values for b_1 thru b_8 : $b_1 = -18$, $b_4 = 12$, $b_6 = 1$, $and\, b_i = 0$ if i $\neq$ 2,4,6. This choise on parameters b_1, ..., b_8 is only one possibility and it is not relevant to the subject of this paper.

The tensor of matter M^i_{jkl} would be defined as a combination (sum) of covariant derivatives (using g_{ij}) of P^i_{jk} ($P^i_{jk;l}$) and the square of P^i_{jk}. Such structure of the tensor of matter M^i_{jkl} is dictated by the scaling factors of P^i_{jk} and M^i_{jkl}: P^i_{jk} is cm^{-1} and M^i_{jkl} is cm^{-2}. Examples of such a tensor matter are shown below:

$$\begin{aligned}
&a) M^i_{jkl} = P^i_{jk;l} \\
&b) M^i_{jkl} = 3P^i_{jk;l} + P^i_{mk}P^m_{jl} \\
&c) M^i_{jkl} = P^i_{jl;k} - P^i_{jk;l} + P^i_{mk}P^m_{jl} - P^i_{ml}P^m_{jk}
\end{aligned} \tag{2}$$

The equation for the distribution of matter is obtained from the variational principle. The Lagrangian depends on tensors M^i_{jkl} and g_{ij}, which both are functions of only the tensor potential P^i_{jk}.

$$A = \int L(M^i_{jkl}(P^i_{jk}, P^i_{jk;l}),\, g_{ij}(P^i_{jk}))\, \sqrt{g(P^i_{jk})}\; d^4x \tag{3}$$

where $g(P^i_{jk}) = -det(g_{ij})$.

By analogy with other flat space field theories, we will assume that the Lagrangian is quadratic with respect to the matter tensor M. From a physical point of view it is more convenient to consider P^i_{jk} and g_{ij} as independent variables. By introducing Lagrange coefficients (or Lagrange multipliers) T^{ij}, we can rewrite the action integral A in this form:

$$A = \int \sqrt{g}\, d^4x\, \{L(P^i_{jk;l}, P^i_{jk}, g_{ij}) + T^{mn}[g_{mn} - X_{mn}(P^i_{jk})]\} \tag{4}$$

Here, $X_{mn}(P^i_{jk})$ is a quadratic function given by eq.(1). The variational principle with respect to P^i_{jk}, g_{ij} and T^{ij} yields the following three sets of Euler equations (for details see [1]):

$$\frac{\delta A}{\delta P^i_{jk}} = 0 \quad or \quad \partial L/\partial P^i_{jk} - (\partial L/\partial P^i_{jk;l})_{;l} - T^{mn}[\partial X_{mn}/\partial P^i_{jk}] = 0 \tag{5}$$

$$\frac{\delta A}{\delta g_{ij}} = 0 \quad or \quad \frac{\delta L}{\delta g_{ij}} + T^{ij} = 0 \tag{6}$$

$$\frac{\delta A}{\delta T^{ij}} = 0 \quad or \quad g_{ij} - X_{ij}(P^i_{jk}) = 0 \tag{7}$$

Obviously, the set of Lagrange multipliers T^{ij}, defined by eq. (6), is the energy-momentum tensor of the matter.

We now show that if the first part of Lagrangian (function L) has the U(1) gauge of the type

$$P^i_{jk} -> P^i_{jk} + a\delta^i_j \phi_{,k} + b\delta^i_k \phi_{,j} + c g_{jk} g^{mi} \phi_{,m} \tag{8}$$

where a,b,c are constants, then the Euler equations of motion (5) contain the covariant low of conservation in the form $J^k{}_{;k} = 0$.

Let us first consider the action without the "constraint" part

$$A = \int \sqrt{g}\, d^4x\ L(P^i_{jk;l}, P^i_{jk}, g_{ij}) \tag{9}$$

Here P^i_{jk} are the independent variables and g_{ij} are some fixed functions not depending on P^i_{jk} The variation of this integral with respect to the tensor P^i_{jk} can be written in this form:

$$\delta A = \int \sqrt{g}\, d^4x\, Q^{jk}_i\, \delta P^i_{jk} \tag{10}$$

where

$$Q^{jk}_i = \partial L/\partial P^i_{jk} - (\partial L/\partial P^i_{jk;l})_{;\,l} \tag{11}$$

Let us now assume that the Lagrangian L has a guage. That is to say, the Lagrangian does not change when we replace P^i_{jk} with $P^i_{jk} + a\delta^i_j \phi_{,k} + b\delta^i_k \phi_{,j} + c g_{jk} g^{mi} \phi_{,m}$,where a,b,c are constants. If we assume that δP^i_{jk} is due to change in fuction ϕ only, we can now rewrite the variation of action integral in this form:

$$\begin{aligned}
\delta A = \int \sqrt{g}\, d^4x\, Q^{jk}_i\, \delta P^i_{jk} = \\
\int \sqrt{g}\, d^4x\, Q^{jk}_i\, \delta(a\delta^i_j \phi_{,k} + b\delta^i_k \phi_{,j} + c g_{jk} g^{mi} \phi_{,m}) = \\
\int \sqrt{g}\, d^4x\, (-a Q^{ik}_i{}_{;k} - b Q^{ki}_i{}_{;k} - c(Q^{ij}_k g_{ij})^{;k})\delta\phi
\end{aligned} \tag{12}$$

Since L has the gauge and thus it does not depend on ϕ, δA must equal zero for any ϕ. From this it follows that $(-aQ_{i\ ;k}^{ik} - bQ_{i\ ;k}^{ki} - c(Q_k^{ij}g_{ij})^{;k}) = 0$ for any P_{jk}^i or, in other words, it is an identity. This in fact is a particular case of the second Noether theorem. The additional illustraton can be found in Appendix A, where we give an example of Lagrangian, full derivation of Euler equations and derivation from them the Noether identities.

We now consider a situation were g_{ij} is not a fixed function, but a function of P_{jk}^i. By introducing Lagrange multipliers T_{ij} (Lagrange coefficient), we can consider g_{ij} to be an independent variable with the Lagrangian represented by (4).

The Euler equation with respect to variation of P_{jk}^i will take this form.

$$
\begin{aligned}
&Q_i^{jk} + J_i^{jk} = 0 \quad where \\
&Q_i^{jk} = \partial L/\partial P_{jk}^i - (\partial L/\partial P_{jk;l}^i)_{;\,l} \quad and \\
&J_i^{jk} = T^{mn}[\partial Q_{mn}/\partial P_{jk}^i]
\end{aligned} \tag{13}
$$

If we now construct the invariant the same way as it appears in (12) we will get:

$$
[(-aQ_{i\ ;k}^{ik} - bQ_{i\ ;k}^{ki} - cQ_k^{ij;k}g_{ij})] + [(-aJ_{i\ ;k}^{ik} - bJ_{i\ ;k}^{ki} - cJ_k^{ij;k}g_{ij})] = 0 \tag{14}
$$

The expression in the first square bracket, as it has been shown above, vanishes leaving the remaining equation in the form:

$$
J_{;k}^k = 0 \quad where \quad J^k = (aJ_i^{ik} + bJ_i^{ki} + cJ_m^{ij}g_{ij}g^{mk}) \tag{15}
$$

This is the covariant law of conservation. Thus we have shown that if Lagrangian has the guage and the constraints on g_{ij} do not (thus semi-gauge), the Euler equations contain the Law of Conservation. The requirement that tensor M_{jkl}^i, which makes up the Lagrangian has this gauge is not necessary as long as Lagrangian by itself has such a gauge.

This situation is very similar to the law of conservation of electrical 4-current in the classical Maxwell electromagnetic theory. The Lagrangian in this case can be written as:

$$I = \int \sqrt{g}\, d^4x\, [F^{ij}F_{ij} + A_i J^i] \quad , where F_{ij} = A_{i;j} - A_{j;i} \tag{16}$$

This Lagrangian has semi-gauge $A_i = A_i + f_{,i}$. This transformation does not change the electromagnetic tensor F_{ij} and thus the first term of the Lagrangian. However, $A_i = A_i + f_{,i}$ is not a gauge for the second term containing the 4-current J^i. As the result, the Maxwell equations contain the law of conservation of 4-current.

It can be seen from eq.(13) that the 4-current J_k in our theory is proportional to T_{ij} or energy-momentum tensor. Using (1) J_k can be written directly in terms of T_{ij} and tensor potential P^i_{jk}. It will contain only 7 terms as shown below:

$$\begin{aligned} J_k = {} & \bar{b}_1 T^{mn} \bar{P}_{mkn} + \bar{b}_2 T^{mn} \hat{P}_{mkn} + \bar{b}_3 T^{mn} \bar{P}_{kmn} \\ & + \bar{b}_4 T^m_k \bar{P}_m + \bar{b}_5 T^m_k \hat{P}_m + \bar{b}_6 T \bar{P} k + \bar{b}_7 T \hat{P} k \end{aligned} \tag{17}$$

Here $\bar{b}_1$ ... $\bar{b}_8$ are constants that depend on constants b_1 ... b_8 of eq.(13) and gauge constant a,b,c.

In [1], we showed that contraction of Euler equations obtained by variation of g_{ij} ($g^{ij}\delta L/\delta g_{ij}$) leads to the equation:

$$\bar{J}^k_{;k} + T = 0 \quad where \quad T = T^i_i \tag{18}$$

This is a direct consequence of the fact that Lagrangian is a quadratic function of the tensor of matter M^i_{jkl}. In order to yield the law of conservation, we had to postulate that for all physically meaningful solutions the invariant T (the trace of T_{ij}) for entire system (including gravitational field) is zero. If the Lagrangian has a semi-gauge, this requirement is no longer needed. However, if in some case T=0, we would have one more 4-current that is conserved.

Lagrangian

We now discuss the question of the freedom that we have in the choice of Lagrangian with such semi-gauge. It is well known that if the tensor $M^i{}_{jkm}$ is defined by expression (2c), the Lagrangian (3) satisfies the gauge of $P^i_{jk} -> P^i_{jk} + \delta^i_j \phi_{,k}$. We will show now that the requirement for Lagrangian having the more general gauge (8) $P^i_{jk} -> P^i_{jk} + a\delta^i_j \phi_{,k} + b\delta^i_k \phi_{,j} + c g_{jk} g^{mi} \phi_{,m}$ is not that difficult to satisfy.

Let us consider Lagrangians that depend only on $P^i_{jk;m}$. In general there are total of 24 (4!) terms in which 4 indices of one tensor P are contracted with 4 indices of the other tensor P. Here are a couple of examples of such terms: $P_{ijkm}P^{ijkm}$ or $P_{ijkm}P^{mijk}$ - where $P_{ijkm} = P_{ijk;m} = g_{is}P^s_{jk;l}$ In the expression above, as in all similar expressions below, the sign of semicolon (;) is dropped. Only 17 out of 24 quadratic scalars are independent. Below we give a complete list of these invariants. The first group represents the terms that are symmetrical with respect to renaming indeces. Meaning, if we rename the indices of the second P to be "ijkm" , the term transfers into itself. For example:

$$P^{ijkm}P_{mkji} =_{(m->i,k->j,j->k,i->m} = P^{mkji}P_{ijkm} = P^{ijkm}P_{mkji} \tag{19}$$

There are 10 such terms:

$$\begin{gathered} P_{ijkm}P^{ijkm},\ P_{ijkm}P^{ijmk},\ P_{ijkm}P^{imkj},\ P_{ijkm}P^{ikjm},\ P_{ijkm}P^{jikm}, \\ P_{ijkm}P^{jimk},\ P_{ijkm}P^{kmij},\ P_{ijkm}P^{kjim},\ P_{ijkm}P^{mjki},\ P_{ijkm}P^{mkji} \end{gathered} \tag{20}$$

The second group contains 14 terms in which only 7 are independent with respect to renaming indices.

$$\begin{aligned} P_{ijkm}P^{ikmj} &=_{(i->i\,,k->j\,,m->k\,,j->m)} = P_{imjk}P^{ijkm} = P_{ijkm}P^{imjk} \\ P_{ijkm}P^{jkmi} &=_{(j->i\,,k->j\,,m->k\,,i->m)} = P_{mijk}P^{ijkm} = P_{ijkm}P^{mijk} \\ P_{ijkm}P^{jmik} &=_{(j->i\,,m->j\,,i->k\,,k->m)} = P_{kimj}P^{ijkm} = P_{ijkm}P^{kimj} \\ P_{ijkm}P^{jmki} &=_{(j->i\,,m->j\,,k->k\,,i->m)} = P_{mikj}P^{ijkm} = P_{ijkm}P^{mikj} \\ P_{ijkm}P^{jkim} &=_{(j->i\,,k->j\,,i->k\,,m->m)} = P_{kijm}P^{ijkm} = P_{ijkm}P^{kijm} \\ P_{ijkm}P^{kjim} &=_{(k->i\,,j->j\,,i->k\,,m->m)} = P_{kjim}P^{ijkm} = P_{ijkm}P^{kjim} \\ P_{ijkm}P^{kmji} &=_{(k->i\,,m->j\,,j->k\,,i->m)} = P_{mkij}P^{ijkm} = P_{ijkm}P^{mkij} \end{aligned} \tag{21}$$

In addition to the terms discussed above, there are terms where the contraction of indices occurs once inside P_{ijkm}. An example of one such term is $P^i{}_{ikm}P_j{}^{jkm}$

In general, the contraction of tensor P_{ijkm} will produce six 2-index tensors: $P^{(1)}_{km} = P^i{}_{ikm}$, $P^{(2)}_{jm} = P^i{}_{jim}$, $P^{(3)}_{jk} = P^i{}_{jki}$, $P^{(4)}_{im} = P_{ijkm}g^{jk}$, $P^{(5)}_{ik} = P_{ijkm}g^{jm}$, $P^{(6)}_{ij} = P_{ijkm}g^{km}$. If the symmetry of these tensors is not defined (meaning, the tensor has both symmetric and antisymmetric parts), there would be 21 Lagrangian terms that can be constructed. These Lagrangian terms could be written in this form:

$$P^{(\alpha)}_{km}P^{(\beta)km}\,,\; P^{(\alpha)}_{km}P^{(\beta)mk} \tag{22}$$

where α and β takes values from 1 to 6 and $\alpha >= \beta$

The six 2-index tensors $P^{(\alpha)}_{km}$ $(\alpha = 1, ..., 6)$ allow to form 3 scalars: $I_1 = P^i{}_{ik}{}^k$, $I_2 = P^i{}_{ki}{}^k$, $I_3 = P^i{}_k{}^k{}_i$. Using these 3 scalars, 6 more Lagrangian terms can be written as a product of the 2 scalers:

$$I_1I_1\,,\; I_1I_2\,,\; I_1I_3\,,\; I_2I_2\,,\; I_2I_3\,,\; I_3I_3 \tag{23}$$

Combining these three types of Lagrangian terms together (eq.(20) thru eq.(23)), we will have the total of 44 terms or 44 unknown constants (out of which, of course, only 43 are independent).

In order to evaluate the ability of Lagrangian to have the gauge of type (1), we have to replace $P^i{}_{jk}$ with $P^i{}_{jk} + a\delta^i_j\phi_{,k} + b\delta^i_k\phi_{,j} + cg_{jk}g^{mi}\phi_{,m}$ and then evaluate the requirement on these 44 constants to assure that the final Lagrangian does not contain any terms that include function ϕ. It is not diifcult to see that due to the symmetry of $\phi_{;i;j}$ $(\phi_{;i;j} = \phi_{;j;i})$ there are only 11 such terms.

$$P^{(\alpha)km}\phi_{;k;m} \quad \alpha = 1\,, \ldots\,, 6 \tag{24}$$

$$I_1\phi_{;k;m}g^{km}\,,\; I_2\phi_{;k;m}g^{km}\,,\; I_3\phi_{;k;m}g^{km} \tag{25}$$

$$\phi_{;i;j}\phi_{;k;m}g^{ik}g^{jm} \; and \; \phi_{;i;j}\phi_{;k;m}g^{ij}g^{km} \tag{26}$$

Thus for any pre-set values of constants a,b,c in the gauge (8) we will have 11 linear equations with 44 unknown constants associated with all possible Lagrangian terms.

The reason why the number of equations is much less then the number of possible Lagrangianian terms is due to the fact that the gauge contains the metric g_{ij}. The presence of the metric tensor reduces the rank of $P^i{}_{jk}$ tensor and thus the number of possible terms that contain gauge function ϕ.

The same can be observed for the other types of Lagrangianian terms - those that consist of $P^i{}_{jk;m}$ and the square of $P^i{}_{jk}$ (an example of such a term could be $P^i{}_{jk;m}P^n{}_{np}P^j{}_{iq}g^{kp}g^{mq}$) or the ones that consist of four-product of $P^i{}_{jk}$ (example - $P_{ijk}P^{imn}P^{jsk}P^s{}_{ms}g_{ns}$).

Thus we conclude that the requirement of Lagrangian to have a gauge of type (8) is quite possible to satisfy.

Appendix A

Here we will illustrate how semi-gauge leads to the law of conservation using a particular example. Let's assume that M^i_{jkl} is defined in terms of P^i_{jk} by formula (2c):

$$M^i_{jkl} = P^i_{jl;k} - P^i_{jk;l} + P^i_{mk}P^m_{jl} - P^i_{ml}P^m_{jk} \tag{27}$$

It is not difficult to show that M^i_{jkl} has a guage with respect to the following transformation of $P^i_{jk} -> P^i_{jk} + \delta^i_j\phi_{,k}$

Since M^i_{jkl} has a gauge, so will a Lagrangian that depends on M^i_{jkl} only. For example if we take $L = M^i_{jkl} * M^j_{imn}g^{km}g^{ln}$, the expression for Euler equations can be written as this:

$$\begin{aligned} &Q_i^{jk} = 0 \quad , \mathit{where} \\ &Q_i{}^{jk} = -2[M_i{}^{jkl}{}_{;l} + M_i^{nkl}P^j{}_{nl} - M_n{}^{jkl}P^n{}_{il}] \end{aligned} \tag{28}$$

Contracting first 2 indices we will have

$$Q_i{}^{ik} = -2[M_i{}^{ikl}{}_{;l} + M_i^{nkl}P^i{}_{nl} - M_n{}^{ikl}P^n{}_{il}] = N^{kl}{}_{;l} \tag{29}$$

where $N^{kl} = M_i^{ikl}$ is antisymmetric tensor due to the antisymmetry of M_i^{jkl} in indices k,l. If we take the covariant derivative of Q_i^{ik} and contract it by index k ($Q_i{}^{ik}{}_{;k}$), we will get:

$$Q_i{}^{ik}{}_{;k} = N^{km}{}_{;k;m} = 1/2(N^{km}{}_{;m;k} - N^{km}{}_{;k;m}) =$$
$$1/2(R_{kms}{}^{k} N^{sm} + R_{kms}{}^{m} N^{ms}) = R_{ms} N^{sm} \equiv 0 \tag{30}$$

The last expression is an identity since tensor Ricci (R_{ij}) is symmetric and tensor N_{ij} is antisymmeric.

Again this fact of identity is direct consequence of second Noether theorem and is true not only for this example, but for any case when Lagrangian has a gauge.

References

[1] B. Hikin, Tensor Potential Description of Matter and Space (http://arxiv.org/list/gr-qc/0508, 043)